This document is the online-only appendix to:

Generalizing Consistency and other Constraint Properties to Quantified Constraints

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A. PROOFS OF THE MAIN PROPOSITIONS

Proposition 1. *A QCSP is true (as defined in Section 2.1.2) iff it has a winning strategy.*

Proof. Instead of proving this result from scratch we sketch its connection to classical logical results and simply note that the functions used in the definition of the notion of strategy are essentially Skolem functions: it is well-known that, starting from a formula $\forall x_1 \dots x_n . \exists y.\ F(x_1, \dots x_n, y)$ with an existentially quantified variable y, we can replace y by a function and obtain a second-order formula that is equivalent: $\exists f.\ \forall x_1 \dots x_n.\ F(x_1, \dots x_n, f(x_1 \dots x_n))$.

If the domain $\mathbb{D}$ is additionally fixed and each quantifier is bounded, i.e., if we have a formula of the form: $\forall x_1 \in D_{x_1} \dots \forall x_n \in D_{x_n} . \exists y \in D_y.\ F(x_1, \dots x_n, y)$, then the formula is equivalent to:

$$\exists f.\ \forall x_1 \in D_{x_1} \dots \forall x_n \in D_{x_n} . (f(x_1 \dots x_n) \in D_y \wedge\ F(x_1, \dots x_n, f(x_1 \dots x_n)))$$

and any interpretation I verifying:

$$\langle \mathbb{D}, I \rangle \models \forall x_1 \in D_{x_1} \dots \forall x_n \in D_{x_n} . (f(x_1 \dots x_n) \in D_y \wedge\ F(x_1, \dots x_n, f(x_1 \dots x_n)))$$

is such that the function $I(f)$ is of signature $\left(\prod_{x_i \in \{x_1 \dots x_n\}} D_{x_i}\right) \rightarrow D_y$.

Now given a QCSP, let F be its logical representation as defined in Section 2.1.2, and let F' be the Skolem normal form of F, obtained by iteratively applying the process described above, for all existential variables. The strategies of the QCSP are exactly the possible interpretations of the Skolem functions of F'. Furthermore, a strategy is winning (all outcomes are true) iff the first-order (universally quantified) part of the formula is true. Consequently a winning strategy exists for the QCSP iff the model-checking problem $\langle \mathbb{D}, I\rangle \models F'$ is true, i.e., iff the QCSP is true. $\square$

PROPOSITION 2. *Deep fixability could equivalently be defined by the condition* $\forall t \in \textsf{out}. t[x_i := a] \in \textsf{sol}$*; Deep substitutability could be equivalently defined by* $\forall t \in \textsf{out}.\ (t_{x_i} = a) \rightarrow (t[x_i := b] \in \textsf{sol})$*; deep removability by* $\forall t \in \textsf{out}.(t_{x_i} = a) \rightarrow (\exists b \neq a. t[x_i := b] \in \textsf{sol})$*; and deep irrelevance by* $\forall t \in \textsf{out}. \forall b \in D_{x_i}.\ t[x_i := b] \in \textsf{sol}$*.*

PROOF. We consider fixability and we prove that $\forall t \in \textsf{out}.\ t[x_i := a] \in \textsf{out}$ holds iff $\forall t \in \textsf{out}.\ t[x_i := a] \in \underline{\textsf{sol}}$ does. The $\rightarrow$ implication is straightforward ($\textsf{out} \subseteq \textsf{sol}$); we prove the $\leftarrow$ implication. In the case where the QCSP is false (no winning strategy) the implication trivially holds, since out is then empty. Let us therefore prove it in the case where the QCSP is true.

We assume that $\forall t \in \textsf{out}.\ t[x_i := a] \in \textsf{sol}$. Let $t \in \textsf{out}$; it is clear that the tuple $t[x_i := a]$ belongs to sol; we have to prove that $t[x_i := a]$ also belongs to out. For that purpose, we exhibit a winning strategy s such that $t[x_i := a] \in \textsf{sce}(s)$.

Let s' be a winning strategy such that $t \in \textsf{sce}(s')$. Such a strategy exists since t is an outcome. The strategy s will be obtained by modifying s' so that all its outcomes assign value a to variable x_i. More formally, the functions s_{x_j} are defined, for each $x_j \in E$, as follows:

—If $j = i$ then $s_{x_j}(\tau) \doteq a$, for each tuple $\tau \in \prod_{y \in A_{j-1}} D_y$;

—Otherwise s_{x_j} is simply defined as the function s'_{x_j}.

One can now verify that $\textsf{sce}(s) = \{\tau[x_i := a]\ :\ \tau \in \textsf{sce}(s')\}$. Two consequences are $t[x_i := a] \in \textsf{sce}(s)$, and $\textsf{sce}(s) \subseteq \textsf{sol}$, which show that s is a winning strategy such that $t[x_i := a] \in \textsf{sce}(s)$.

Similarly, for substitutability we can exhibit a strategy s in which every $t \in \textsf{sce}(s')$ such that $t_{x_i} = a$ is changed into the scenario $t[x_i := b]$.

For removability it is convenient to restate the property: removability holds if there exists a function f that associates to every X-tuple t a value $f(t) \neq a$, and such that $\forall t \in \textsf{out}.(t_{x_i} = a) \rightarrow (t[x_i := f(t)] \in \textsf{out})$. We can exhibit a strategy s in which every $t \in \textsf{sce}(s)$ such that $t_{x_i} = a$ is changed into the scenario $t[x_i := f(t)]$.

For irrelevance we can use the fact that a variable is irrelevant iff it can be fixed to any value of its domain (Prop. 5). $\square$

PROPOSITION 3. *Let* $\phi = \langle X, Q, D, C\rangle$ *be a QCSP and let* ψ *be the same QCSP but in which all quantifiers are existential, i.e.,* $\psi = \langle X, Q', D, C\rangle$*, with* $Q'_x = \exists$*, for all* $x \in X$*. We have (forall* x_i, a, b, V*):*

—*inconsistent*$^{\psi}(x_i, a) \rightarrow$ *inconsistent*$^{\phi}(x_i, a)$*;*

—*d-fixable*$^{\psi}(x_i, a) \rightarrow$ *d-fixable*$^{\phi}(x_i, a)$*;*

—*d-substitutable*$^{\psi}(x_i, a, b) \rightarrow$ *d-substitutable*$^{\phi}(x_i, a, b)$*;*

—*d-removable*$^{\psi}(x_i, a) \rightarrow$ *d-removable*$^{\phi}(x_i, a)$*;*

—*d-interchangeable*$^{\psi}(x_i, a, b) \rightarrow$ *d-interchangeable*$^{\phi}(x_i, a, b)$*;*

—*determined*$^{\psi}(x_i) \rightarrow$ *determined*$^{\phi}(x_i)$*;*

—*d-irrelevant*$^{\psi}(x_i) \rightarrow$ *d-irrelevant*$^{\phi}(x_i)$*;*

—*dependent*$^{\psi}(V, x_i) \rightarrow$ *dependent*$^{\phi}(V, x_i)$*.*

PROOF. All the results rely essentially on the fact that $\mathsf{out} \subseteq \mathsf{sol}$. For the properties of inconsistency, implication, determinacy and dependence, the proof directly follows: classical inconsistency means that $\forall t \in \mathsf{sol}.\ t_{x_i} \neq a$, which implies the deep property $\forall t \in \mathsf{out}.\ t_{x_i} \neq a$; classical determinacy means that $\forall t \in \mathsf{sol}.\ \forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{sol}$, which implies $\forall t \in \mathsf{out}.\ \forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{sol}$, which implies the deep property $\forall t \in \mathsf{out}.\ \forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{out}$. The cases of implication and dependence are similar.

For the other properties we additionally use Proposition 2: classical fixability means that $\forall t \in \mathsf{sol}.\ t[x_i := a] \in \mathsf{sol}$. This implies $\forall t \in \mathsf{out}.\ t[x_i := a] \in \mathsf{sol}$ which, by Proposition 2, is equivalent to the deep property $\forall t \in \mathsf{out}.\ t[x_i := a] \in \mathsf{out}$. The cases of substitutability, removability, interchangeability and irrelevance are similar. □

PROPOSITION 4. *For all variables x_i and values a and b, we have:*

—*d-fixable*$(x_i, a) \rightarrow$ *s-fixable*(x_i, a);
—*d-removable*$(x_i, a) \rightarrow$ *s-removable*(x_i, a);
—*d-substitutable*$(x_i, a, b) \rightarrow$ *s-substitutable*(x_i, a, b);
—*d-interchangeable*$(x_i, a, b) \rightarrow$ *s-interchangeable*(x_i, a, b);
—*d-irrelevant*$(x_i) \rightarrow$ *s-irrelevant*(x_i).

PROOF. If deep fixability holds, i.e., we have $\forall t \in \mathsf{out}.\ t[x_i := a] \in \mathsf{out}$, then for each $t \in \mathsf{out}$ the tuple $t' = t[x_i := a]$ is such that $t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = a$, and we therefore have $\forall t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = a)$, which means *s-fixable*(x_i, a). The proof is similar for irrelevance.

If deep removability holds, i.e., $\forall t \in \mathsf{out}.\ (t_{x_i} = a) \rightarrow (\exists b \neq a.\ t[x_i := b] \in \mathsf{out})$, then for each $t \in \mathsf{out}$ such that $t_{x_i} = a$, the tuple $t' = t[x_i := b]$ is such that $t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = b$, and we have *s-removable*(x_i, a). The proof is similar for substitutability, which also uses a bounded quantification, and the result follows for interchangeability. □

PROPOSITION 5. *The following relations hold between the properties (forall x_i, a and b):*

(1) *inconsistent*$(x_i, a) \rightarrow \forall b \in D_{x_i}.$ *d-substitutable*(x_i, a, b);
(2) *implied*$(x_i, a) \leftrightarrow \forall b \in D_{x_i} \setminus \{a\}.$ *inconsistent*(x_i, b);
(3) *implied*$(x_i, a) \rightarrow$ *d-fixable*(x_i, a);
(4) *inconsistent*$(x_i, a) \rightarrow$ *d-removable*(x_i, a);
(5) $\exists b \in D_{x_i} \setminus \{a\}.$ *d-substitutable*$(x_i, a, b) \rightarrow$ *d-removable*(x_i, a);
(6) $\exists b \in D_{x_i} \setminus \{a\}.$ *s-substitutable*$(x_i, a, b) \rightarrow$ *s-removable*(x_i, a);
(7) *d-fixable*$(x_i, b) \leftrightarrow \forall a \in D_{x_i}.$ *d-substitutable*(x_i, a, b);
(8) *s-fixable*$(x_i, b) \leftrightarrow \forall a \in D_{x_i}.$ *s-substitutable*(x_i, a, b);
(9) *d-irrelevant*$(x_i) \leftrightarrow \forall a \in D_{x_i}.$ *d-fixable*(x_i, a);
(10) *s-irrelevant*$(x_i) \leftrightarrow \forall a \in D_{x_i}.$ *s-fixable*(x_i, a).

PROOF. (1) Assume inconsistency holds. If we consider an arbitrary $t \in \mathsf{out}$, then $t_{x_i} \neq a$, which falsifies the left side of the implication $(t_{x_i} = a) \rightarrow (t[x_i := b] \in \mathsf{out})$, for any b, and deep substitutability therefore holds.

(2) If value a is implied for x_i, i.e., $\forall t \in \mathsf{out}.\ t_{x_i} = a$, then for every value $b \neq a$ we have $\forall t \in \mathsf{out}.\ t_{x_i} = a \neq b$, i.e., b is inconsistent. If all values $b \neq a$ are inconsistent, i.e., $\forall t \in \mathsf{out}.\ t_{x_i} \neq b$, then any $t \in \mathsf{out}$ is such that $\forall b \neq a.\ t_{x_i} \neq b$ and $t_{x_i} \in D_{x_i}$, so $t_{x_i} = a$ i.e., a is implied.

(3) If a is implied for x_i, then any $t \in \mathsf{out}$ is such that $t_{x_i} = a$, and we therefore have $t[x_i := a] = t \in \mathsf{out}$.

(4) If a is inconsistent for x_i, i.e., $\forall t \in \mathsf{out}.\ t_{x_i} \neq a$, then the left-hand side of the implication $(t_{x_i} = a) \rightarrow (\exists b \neq a.\ t[x_i := b] \in \mathsf{out})$ is false for every $t \in \mathsf{out}$.

(5) If a is deep-substitutable to a certain value $b \neq a$, then for every $t \in \mathsf{out}$ verifying $t_{x_i} = a$ we have $t[x_i := b] \in \mathsf{out}$. This implies $\exists b \neq a.\ t[x_i := b] \in \mathsf{out}$.

(6) If a is shallow-substitutable to a certain value $b \neq a$, then for every $t \in \mathsf{out}$ verifying $t_{x_i} = a$, we have $\exists t' \in \mathsf{out}.\ ((t|_{X_{i-1}} = t'|_{X_{i-1}}) \wedge (t'_{x_i} = b))$. This implies $\exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} \neq a)$.

(7) If b is deep-fixable for x_i, i.e., $\forall t \in \mathsf{out}.\ t[x_i := b] \in \mathsf{out}$, then the right-hand side of the implication $(t_{x_i} = a) \rightarrow (t[x_i := b] \in \mathsf{out})$ is true for all $t \in \mathsf{out}$.

(8) If b is shallow-fixable for x_i i.e., $\forall t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = b)$, then the right-hand side of the implication $t_{x_i} = a \rightarrow \exists t' \in \mathsf{out}.\ ((t|_{X_{i-1}} = t'|_{X_{i-1}}) \wedge (t'_{x_i} = b)$ is true for all $t \in \mathsf{out}$.

(9) If x_i is deep-irrelevant, i.e., $\forall t \in \mathsf{out}.\ \forall a \in D_{x_i}.\ t[x_i := a] \in \mathsf{out}$, then for any $a \in D_{x_i}$ we have $\forall t \in \mathsf{out}.\ t[x_i := a] \in \mathsf{out}$.

(10) If x_i is shallow-irrelevant, i.e., $\forall t \in \mathsf{out}.\ \forall a \in D_{x_i}.\ \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}}) \wedge (t'_{x_i} = a)$, then for any $a \in D_{x_i}$ we have $\forall t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = a)$. $\square$

Proposition 6. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP in which value $a \in D_{x_i}$ is shallow-removable for an existential variable x_i, and let ϕ' denote the same QCSP in which value a is effectively removed (i.e., $\phi' = \langle X, Q, D', C\rangle$ where $D'_{x_i} = D_{x_i} \setminus \{a\}$ and $D'_{x_j} = D_{x_j}, \forall j \neq i$). Then ϕ is true iff ϕ' is true.*

Proof. If ϕ' has a winning strategy then the same strategy is also winning for ϕ; having ϕ' true therefore implies that ϕ is also true.

On the other hand, assume that ϕ has a winning strategy s^1. Since $a \in D_{x_i}$ is shallow-removable for x_i, we have:

$$\forall t \in \mathsf{out}.\ t_{x_i} = a \rightarrow \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} \neq a.)$$

We show that if s^1 has a scenario $t \in \mathsf{sce}(s^1)$ such that $t_{x_i} = a$, then we can "correct" this and exhibit another winning strategy s whose scenarios are the same as those of s^1 except that all scenarios λ such that $\lambda|_{X_{i-1}} = t|_{X_{i-1}}$ have been replaced by tuples t' with $t'_{x_i} \neq a$. (Intuitively we replace the "sub-tree" corresponding to the branch $t|_{X_{i-1}}$ by a new branch which does not involve the choice $x_i = a$ anymore.) More precisely, every scenario $t' \in \mathsf{sce}(s)$ will satisfy:

—If $t'|_{X_{i-1}} \neq t|_{X_{i-1}}$ then $t' \in \mathsf{sce}(s^1)$.
—If $t'|_{X_{i-1}} = t|_{X_{i-1}}$ then $t'_{x_i} \neq a$.

This will prove the result: in showing how to construct s we show that, starting from any winning strategy s^1 containing a number $n > 0$ of "incorrect" scenarios t' with $t'_{x_i} = a$, we can always exhibit a winning strategy with at most $n - 1$ such scenarios, and repeating the correction n times we construct a winning strategy in which no tuple t' is such that $t'_{x_i} = a$.

Let us now see how to construct s starting from s^1. The outcome $t \in \mathsf{sce}(s^1)$ that needs to be replaced is such that $t_{x_i} = a$ and, using the shallow removability property, we conclude that there exists another outcome $\theta \in \mathsf{out}$ such that $\theta|_{X_{i-1}} = t|_{X_{i-1}} \wedge \theta_{x_i} \neq a$. This outcome belongs to at least one winning strategy. We choose one of these strategies, which we call s^2. To define the new strategy s we must define the functions s_{x_j}, for each $x_j \in E$. These functions are defined as follows:

—if $j < i$ then s_{x_j} is defined as $s^1_{x_j}$ (e.g., we follow the strategy s^1 for the first variables, until variable x_i, excluded);

—for the following variables, i.e., when $j \geq i$, we define the value of $s_{x_j}(\tau)$, for each $\tau \in \prod_{y \in A_{j-1}} D_y$, as follows:
  —if $\tau|_{X_{i-1}} = t|_{X_{i-1}}$, then $s_{x_j}(\tau) = s^2_{x_j}(\tau)$;
  —if $\tau|_{X_{i-1}} \neq t|_{X_{i-1}}$, then $s_{x_j}(\tau) = s^1_{x_j}(\tau)$;

The proof is completed by checking that every scenario $t' \in \mathsf{sce}(s)$ satisfies the two desired properties:

—If $t'|_{X_{i-1}} \neq t|_{X_{i-1}}$ then $t' \in \mathsf{sce}(s^1)$, because, for each $x_j \in E$, we have $t'_{x_j} = s_{x_j}(t'|_{A_{j-1}}) = s^1_{x_j}(t'|_{A_{j-1}})$ in this case.

—If $t'|_{X_{i-1}} = t|_{X_{i-1}}$ then $t'_{x_i} \neq a$, because $t'_{x_i} = s_{x_i}(t'|_{A_{i-1}}) = s^2_{x_i}(t'|_{A_{i-1}}) = s^2_{x_i}(t|_{A_{i-1}}) = s^2_{x_i}(\theta|_{A_{i-1}}) = \theta_{x_i} \neq a$.

Furthermore, every $t' \in \mathsf{sce}(s)$ with $t'|_{X_{i-1}} = t|_{X_{i-1}}$ belongs to $\mathsf{sce}(s^2)$, and s is therefore a winning strategy: $\mathsf{sce}(s) \subseteq (\mathsf{sce}(s^1) \cup \mathsf{sce}(s^2)) \subseteq \mathsf{out}$. □

PROPOSITION 7. *Let $\phi = \langle X, Q, D, C \rangle$ be a QCSP in which value $a \in D_{x_i}$ is fixable for an existential variable x_i, and let ϕ' denote the same QCSP in which value a is effectively fixed (i.e., $\phi' = \langle X, Q, D', C \rangle$ where $D'_{x_i} = \{a\}$ and $D'_{x_j} = D_{x_j}, \forall j \neq i$). Then ϕ is true iff ϕ' is true.*

PROOF. If ϕ' has a winning strategy then the same strategy is also winning for ϕ; having ϕ' true therefore implies that ϕ is also true.

On the other hand suppose that ϕ has a winning strategy s^1. That $a \in D_{x_i}$ is shallow-fixable for x_i means that we have:

$$\forall t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ (t|_{X_{i-1}} = t'|_{X_{i-1}} \wedge t'_{x_i} = a)$$

The proof is similar to the one already detailed for Prop. 6: we show that if s^1 has a scenario $t \in \mathsf{sce}(s^1)$ such that $t_{x_i} \neq a$, then we can "correct" this and exhibit another winning strategy s whose scenarios are the same as those of s^1 except that all scenarios λ such that $\lambda|_{X_{i-1}} = t|_{X_{i-1}}$ have been replaced by tuples t' with $t'_{x_i} = a$. More precisely, every scenario $t' \in \mathsf{sce}(s)$ will satisfy:

—If $t'|_{X_{i-1}} \neq t|_{X_{i-1}}$ then $t' \in \mathsf{sce}(s^1)$.

—If $t'|_{X_{i-1}} = t|_{X_{i-1}}$ then $t'_{x_i} = a$.

This will prove the result: in showing how to construct s we show that, starting from any winning strategy s^1 containing a number $n > 0$ of "incorrect" scenarios t' with $t'_{x_i} \neq a$, we can always exhibit a winning strategy with at most $n - 1$ such scenarios. This shows that there exists a winning strategy in which no tuple t' is such that $t'_{x_i} \neq a$.

Let us now see how to construct s starting from s^1. The outcome $t \in \mathsf{sce}(s^1)$ needs to be replaced. Using the shallow fixability property, we know that there exists another outcome $\theta \in \mathsf{out}$ such that $\theta|_{X_{i-1}} = t|_{X_{i-1}} \wedge \theta_{x_i} = a$. This outcome belongs to at least one winning strategy. We choose one of these strategies, which we call s^2. To define the new strategy s we must define the functions s_{x_j}, for each $x_j \in E$. These functions are defined as follows:

—if $j < i$ then s_{x_j} is defined as $s^1_{x_j}$ (e.g., we follow the strategy s^1 for the first variables, until variable x_i, excluded);

—for the following variables, i.e., when $j \geq i$, we define the value of $s_{x_j}(\tau)$, for each $\tau \in \prod_{y \in A_{j-1}} D_y$, as follows:
  —if $\tau|_{X_{i-1}} = t|_{X_{i-1}}$, then $s_{x_j}(\tau) = s^2_{x_j}(\tau)$;
  —if $\tau|_{X_{i-1}} \neq t|_{X_{i-1}}$, then $s_{x_j}(\tau) = s^1_{x_j}(\tau)$;

The proof is completed by checking that every scenario $t' \in \mathsf{sce}(s)$ satisfies the two desired properties:

—If $t'|_{X_{i-1}} \neq t|_{X_{i-1}}$ then $t' \in \mathsf{sce}(s^1)$, because, for each $x_j \in E$, we have $t'_{x_j} = s_{x_j}(t'|_{A_{j-1}}) = s^1_{x_j}(t'|_{A_{j-1}})$ in this case.

—If $t'|_{X_{i-1}} = t|_{X_{i-1}}$ then $t'_{x_i} = a$, because $t'_{x_i} = s_{x_i}(t'|_{A_{i-1}}) = s^2_{x_i}(t'|_{A_{i-1}}) = s^2_{x_i}(t|_{A_{i-1}}) = s^2_{x_i}(\theta|_{A_{i-1}}) = \theta_{x_i} = a$.

Furthermore, every $t' \in \mathsf{sce}(s)$ with $t'|_{X_{i-1}} = t|_{X_{i-1}}$ belongs to $\mathsf{sce}(s^2)$, and s is therefore a winning strategy: $\mathsf{sce}(s) \subseteq (\mathsf{sce}(s^1) \cup \mathsf{sce}(s^2)) \subseteq \mathsf{out}$. $\square$

PROPOSITION 8. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP in which a value $a \in D_{x_i}$ is inconsistent for a universal variable x_i. Then ϕ is false.*

PROOF. Assume that there is a winning strategy s for ϕ. Hence, $\mathsf{sce}(s) \subseteq \mathsf{out}^\phi$. Since x_i is universal, there must exist scenarios t for s such that $t_{x_i} = a$. But, by the definition of inconsistency, $\forall t \in \mathsf{out}^\phi . t_{x_i} \neq a$. Contradiction. $\square$

PROPOSITION 9. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP in which value $a \in D_{x_i}$ is dual-shallow-removable for a universal variable x_i, and let ϕ' denote the same QCSP in which value a is effectively removed (i.e., $\phi' = \langle X, Q, D', C\rangle$ where $D'_{x_i} = D_{x_i} \setminus \{a\}$ and $D'_{x_j} = D_{x_j}, \forall j \neq i$). Then ϕ is true iff ϕ' is true.*

PROOF. Direct consequence of Prop. 6: the hypothesis is that the dual-shallow-removability holds, i.e., a is removable for x_i w.r.t. the negated QCSP $\neg\phi$; then ϕ is true iff $\neg\phi$ is false iff $\neg\phi'$ is false iff ϕ' is true. $\square$

PROPOSITION 10. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP in which value $a \in D_{x_i}$ is dual-shallow-fixable for an universal variable x_i, and let ϕ' denote the same QCSP in which value a is effectively fixed (i.e., $\phi' = \langle X, Q, D', C\rangle$ where $D'_{x_i} = \{a\}$ and $D'_{x_j} = D_{x_j}, \forall j \neq i$). Then ϕ is true iff ϕ' is true.*

PROOF. Direct consequence of Prop. 7: the hypothesis is that the dual-shallow-fixability holds, i.e., a is fixable for x_i w.r.t. the negated QCSP $\neg\phi$; then ϕ is true iff $\neg\phi$ is false iff $\neg\phi'$ is false iff ϕ' is true. $\square$

PROPOSITION 11. *The pure value property allows to detect values that are dual-inconsistent, i.e.,*

$$pure(x_i, a) \;\rightarrow\; dual\text{-}inconsistent(x_i, a)$$

PROOF. The property $pure(x_i, a)$ means that:

$$\mathsf{sol}^{\phi} \;\supseteq\; \{t \in \Pi_{x\in X} D_x \;:\; t_{x_i} = a\}$$

For any set $A \subseteq \Pi_{x\in X} D_x$ we denote by $\overline{A}$ the set of tuples of $\Pi_{x\in X} D_x$ that are not in A. We deduce:

$$\overline{\mathsf{sol}^{\phi}} \;\subseteq\; \overline{\{t \in \Pi_{x\in X} D_x \;:\; t_{x_i} = a\}}$$

which rewrites to:

$$\mathsf{sol}^{\neg\phi} \;\subseteq\; \{t \in \Pi_{x\in X} D_x \;:\; t_{x_i} \neq a\}$$

Furthermore we have $\mathsf{out}^{\neg\phi} \subseteq \mathsf{sol}^{\neg\phi}$. We conclude that

$$\mathsf{out}^{\neg\phi} \;\subseteq\; \{t \in \Pi_{x\in X} D_x \;:\; t_{x_i} \neq a\}$$

In other words any tuple $t \in \mathsf{out}^{\neg\phi}$ is such that $t_{x_i} \neq a$, which is exactly the definition of dual-inconsistency for variable x_i and value a. □

PROPOSITION 12. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP in which value $a \in D_{x_i}$ is deep-fixable for a universal variable x_i, and let ϕ' denote the same QCSP in which value a is effectively* removed *(i.e., $\phi' = \langle X, Q, D', C\rangle$ where $D'_{x_i} = D_{x_i} \setminus \{a\}$ and $D'_{x_j} = D_{x_j}, \forall j \neq i$). If $D'_{x_i} \neq \emptyset$ (i.e., a was not the only element in D_{x_i}), then ϕ is true iff ϕ' is true.*

PROOF. *(ϕ true implies ϕ' true, easy direction)* If ϕ is true, then there exist winning strategies for it. Let s be one such strategy. Given a value $b \in D_{x_i}, b \neq a$ (such b exists by hypothesis), a winning strategy s' for ϕ' can be built by "restricting" s: for each existential variable x_j, s' is defined, for each $\tau \in \Pi_{y\in A_{j-1}} D'_y$ as $s'(\tau) = s(\tau)$.

(ϕ' true implies ϕ true) If ϕ' is true, then there exist winning strategies for it. Let s' be one such strategy. In order to show that ϕ is also true, we will "extend" s' in order to take into account the value a that can be assigned to the universal variable x_i.

Given a value $b \in D'_{x_i}$ (such b exists by hypothesis), a winning strategy s for ϕ can be built from s' as follows:

—For each existential variable x_j, $j < i$, s is defined exactly as s';
—For each existential variable x_j, $j > i$, s is defined, for each $\tau \in \Pi_{y\in A_{j-1}} D_y$ as:
 i) $s(\tau) = s'(\tau)$ if $\tau_{x_i} \neq a$;
 ii) $s(\tau) = s'(\tau[x_i := b])$ if $\tau_{x_i} = a$;

Since s' is winning for ϕ', we have that $\mathsf{sce}(s') \subseteq \mathsf{out}^{\phi'}$; moreover, since by construction, $\mathsf{out}^{\phi'} \subseteq \mathsf{out}^{\phi}$, we have that $\mathsf{sce}(s') \subseteq \mathsf{out}^{\phi}$.

In order to show that s is winning for ϕ, we will characterize the set $\mathsf{sce}(s)$ and prove that $\mathsf{sce}(s) \subseteq \mathsf{out}^{\phi}$.

By the definition of s, we have that the set of its scenarios is contained in the union of two sets:

—$\mathsf{sce}(s')$ (from i))

—$\{t[x_i := a] \mid t \in \mathsf{sce}(s') \;\wedge\; t_{x_i} = b\}$ (from ii))

So, $\mathsf{sce}(s) \subseteq \mathsf{sce}(s') \cup \{t[x_i := a] \mid t \in \mathsf{sce}(s') \;\wedge\; t_{x_i} = b\}$.

As for the first set, $\mathsf{sce}(s') \subseteq \mathsf{out}^\phi$. We now prove that also the second set is a subset of out^ϕ. Assume, by contradiction, that there exists an element in this set that does not belong to out^ϕ. By construction, such element is obtained by a $t \in \mathsf{sce}(s')$ by replacing its x_i-component with a. Since $\mathsf{sce}(s') \subseteq \mathsf{out}^\phi$, we have that $t \in \mathsf{out}^\phi$, and $t[x_i := a] \notin \mathsf{out}^\phi$. Hence a would be not deep-fixable for x_i in ϕ. Contradiction.

So, we have proved that $\mathsf{sce}(s) \subseteq \mathsf{out}^\phi$, hence that s is a winning strategy for ϕ. $\square$

Proposition 13. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP. Given a tuple $t \in \prod_{x\in X} D_x$, we denote by B the conjunction of constraints:*

$$\bigwedge_{x_i \in E} \left(\left(\bigwedge_{y \in A_{i-1}} y = t_y \right) \rightarrow (x_i = t_{x_i}) \right) \tag{4}$$

The QCSP $\psi = \langle X, Q, D, B \cup C\rangle$ is true iff $t \in \mathsf{out}^\phi$.

Proof. Assume that ψ is true. Then it has a non empty set of winning strategies; let s be one of them, picked arbitrarily. Let t' be the scenario of s that is such that $t'|_A = t|_A$, i.e., that assigns the same values as t on the universal variables. Because s is a winning strategy, t' is a solution, and it satisfies the constraint given by (4). A straightforward induction on the indices of the existential variables shows that t is indeed identical to t', which implies $t \in \mathsf{out}^\phi$.

Assume now that $t \in \mathsf{out}^\phi$, i.e., there exists a winning strategy s for ϕ such that $t \in \mathsf{sce}(s)$. Every scenario $t' \in \mathsf{sce}(s)$ satisfies C. Let us prove by case that each $t' \in \mathsf{sce}(s)$ also satisfies B. If we consider the scenario t' which is such that $t'|_A = t|_A$, then this scenario is indeed t (a strategy defines a unique outcome for each assignment of the universal variables), which satisfies B. On the other hand, B is satisfied also if we consider any tuple t' which is such that $t'|_A \neq t|_A$. To see this, let j be the lowest index such that $t'|_{A_{j-1}} \neq t|_{A_{j-1}}$. Constraints of B with $i < j$ are satisfied because $t'|_{A_{i-1}} = t|_{A_{i-1}}$; the others because the left-hand side of the implications $\left(\bigwedge_{y\in A_{i-1}} t'_y = t_y\right) \rightarrow (t'_{x_i} = t_{x_i})$ are false. Every scenario of s therefore satisfies $B \wedge C$, in other words this strategy is winning for ψ. $\square$

Proposition 14. *Given a QCSP $\phi = \langle X, Q, D, C\rangle$, the problems of deciding whether:*

—*value $a \in D_{x_i}$ is* d-fixable, d-removable, inconsistent, implied *for variable $x_i \in X$,*

—*value $a \in D_{x_i}$ is* d-substitutable *to or* d-interchangeable *with $b \in D_{x_i}$ for variable $x_i \in X$,*

—*variable $x_i \in X$ is* dependent *on variables $V \subseteq X$, or is* d-irrelevant*,*

are PSPACE-complete.

PROOF. (membership in PSPACE) The membership in PSPACE relies essentially on Prop. 13 and its immediate consequence, mentioned in the main text, that testing whether $t \in \mathsf{out}$ can be done in polynomial space. All properties hold iff some statement is verified for all $t \in \mathsf{out}$, so the idea is then to loop over each tuple t, determine whether it belongs to out and, if this is the case, check whether it satisfies the statement. For inconsistency we check whether $t_{x_i} \neq a$. We return false as soon as we have a tuple $t \in \mathsf{out}$ for which this is not the case. For implication we test whether $t_{x_i} = a$ and similarly return false if one tuple does not verify that. The same idea works for all properties: for fixability we test whether $t[x_i := a] \in \mathsf{out}$; for substitutability we check whether $(t_{x_i} = a) \rightarrow (t[x_i := b] \in \mathsf{out})$; for removability we check whether $(t_{x_i} = a) \rightarrow (\exists b \neq a.\ t[x_i := b] \in \mathsf{out})$; for determinacy we check whether $\forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{out}$; for irrelevance we check whether $\forall b \in D_{x_i}.\ t[x_i := b] \in \mathsf{out}$. For dependency we have to do a double loop in lexicographical order, check whether both tuples t, t' belong to out and, if, so, check whether $(\forall x_j \in V.\ t_{x_j} = t'_{x_j}) \rightarrow (t_{x_i} = t'_{x_i})$. In any case, at the end of the loop, we return true if no counter-example to the property has been found. It is clear that these algorithms use polynomial space and return true iff the considered property holds. $\square$

PROOF. (hardness for PSPACE) For all properties we reduce the problem of deciding whether a QCSP $\phi = \langle X, Q, D, C\rangle$ is false to the problem of testing whether the considered property holds.

The reductions work as follows. For **inconsistency** we simply construct the QCSP $\psi = \langle X \cup \{x\}, Q', D', C\}\rangle$, where:

—x is a fresh variable, i.e., $x \notin X$;

—Q' is similar to Q except that the new variable x is quantified existentially, i.e., $Q'_y = Q_y,\ \forall y \neq x$ and $Q'_x = \exists$;

—D' is similar to D except that the domain of the new variable x is a singleton, i.e., $D'_y = D_y,\ \forall y \neq x$ and $D'_x = \{a\}$ for some arbitrary a.

It is straightforward that ϕ has a winning strategy iff ψ also does. Let us verify that ϕ is false iff value a is inconsistent for variable x in ψ: if ϕ is false then out^ϕ is empty, and so is out^ψ, and then it is true that $\forall t \in \mathsf{out}^\psi.\ t_{x_i} \neq a$; if a is inconsistent for x in ψ then $\forall t \in \mathsf{out}^\psi.\ t_{x_i} \neq a$, but no outcome can assign a value different from a to variable x_i, hence out^ψ is empty and out^ϕ is also empty.

The same reduction works directly for **removability**: ϕ is false iff a is removable from x in ψ.

For **fixability**, **implication**, **substitutability**, **interchangeability** and **irrelevance**, the reduction is only slightly different; now we construct the QCSP:

$$\psi = \langle X \cup \{x\}, Q', D', C \cup \{x = 0\}\}\rangle$$

in which the new variable x is existential and ranges over $\{0, 1\}$. Note that the constraint $x = 0$ can be expressed directly in each and every of our 5 formalisms. We can check that ϕ is false iff:

—variable x is fixable to value 1 in ψ: if ϕ is false then out^ϕ is empty and so is out^ψ and we trivially have $\forall t \in \mathsf{out}^\psi.\ t[x := 1] \in \mathsf{out}^\psi$; if x is fixable to 1 in ψ

then $\forall t \in \mathsf{out}^{\psi}.\ t[x := 1] \in \mathsf{out}^{\psi}$, but there is no t is such that $t[x := 1] \in \mathsf{out}^{\psi}$ and out^{ψ} and out^{ϕ} are empty.

—value 1 is implied for variable x in ψ: similarly to fixability we have $\mathsf{out}^{\phi} = \emptyset$ iff $\forall t \in \mathsf{out}^{\psi}.\ t_x = 1$.

—value 0 is substitutable to value 1 for variable x in ψ ($\mathsf{out}^{\phi} = \emptyset$ holds iff $\forall t \in \mathsf{out}^{\psi}.\ (t_x = 0) \rightarrow (t[x := 1] \in \mathsf{out}^{\psi})$).

—value 0 is interchangeable with value 1 for variable x in ψ: ($\mathsf{out}^{\phi} = \emptyset$ holds iff $\forall t \in \mathsf{out}^{\psi}.\ (t_x = 0) \leftrightarrow (t[x := 1] \in \mathsf{out}^{\psi})$).

—variable x is irrelevant in ψ: if $\forall t \in \mathsf{out}^{\psi}.\ \forall b \in \{0,1\}.\ t[x := b] \in \mathsf{out}^{\psi}$, then any $t \in \mathsf{out}^{\psi}$ is in particular such that $t[x := 1] \in \mathsf{out}^{\psi}$ so no such t exists and $\mathsf{out}^{\psi} = \emptyset$. (The other direction is trivial.)

For **determinacy** and **dependence**, the reduction consists in constructing the QCSP $\psi = \langle X \cup \{x\}, Q', D', C\}\rangle$, in which the new variable x is existential and ranges over $\{0,1\}$.

We check that ϕ is false if x is determined in ψ. Assume that $\forall t \in \mathsf{out}^{\psi}.\ \forall b \neq t_x.\ t[x := b] \notin \mathsf{out}^{\psi}$, and let us consider an arbitrary $t \in \mathsf{out}^{\psi}$. Its value on x is either 0 or 1 (say 0). Then it is such that $t[x := 1] \notin \mathsf{out}^{\psi}$. Because values 0 and 1 play a symmetric role, this cannot be, and $\mathsf{out}^{\phi} = \emptyset$. (The other implication is trivial.)

We last check that ϕ is false if variable x is dependent on the set of variables X in ψ. Assume that $\forall t, t' \in \mathsf{out}.\ (t|_X = t'|_X) \rightarrow (t_x = t'_x)$. Let us consider an arbitrary tuple $t \in \mathsf{out}^{\psi}$ with (say) $t_x = 0$. If we consider the tuple $t' = t[x := 1]$, then this tuple is such that $t'|_X = t|_X$, and therefore does not belong to out^{ψ} (if it did, then we'd have $t'_x = t_x$). Because values 0 and 1 play a symmetric role, this cannot be, and $\mathsf{out}^{\phi} = \emptyset$. (The other implication is trivial.)

In all our reductions, we can start from any of the 5 formalisms listed in Sec. 6.1, and the resulting QCSP is expressed in the same formalism. It is well-known that deciding the truth of a QCSP in any of these formalisms is PSPACE-complete and the hardness result therefore holds in all 5 cases. □

PROPOSITION 15. *Given a QCSP $\phi = \langle X, Q, D, C\rangle$, the problems of deciding whether:*

—*value $a \in D_{x_i}$ is s-fixable, s-removable for variable $x_i \in X$,*

—*value $a \in D_{x_i}$ is s-substitutable to or s-interchangeable with $b \in D_{x_i}$ for variable $x_i \in X$,*

—*variable $x_i \in X$ is s-irrelevant,*

are PSPACE-complete.

PROOF. For membership in PSPACE the algorithm is similar to Prop. 14: we use the fact that testing whether $t \in \mathsf{out}$ can be done in polynomial space by Prop. 13. To check whether a property of the form $\forall t \in \mathsf{out}.\ \gamma$ is true, we loop over all tuples in lexicographical order, test whether the current tuple is an outcome and, if so, verify that it satisfies γ. For properties of the form $\exists t \in \mathsf{out}.\ \gamma$, we do a similar loop and return true iff one of the outcomes met during the loop satisfied γ. This works in polynomial space for all properties.

The hardness is a direct consequence of the fact that shallow properties are equivalent to the deep ones in the particular case when the variable on which the property is asserted is at the tail of the linearly ordered set of variables. In all the reductions used in the proof of Prop. 14, note that we introduce a variable that can be introduced *at an arbitrary place*. The reductions can therefore be directly adapted to the shallow definitions.

For instance, in the case of fixability, the reduction consisted, starting from a QCSP $\phi = \langle X, Q, D, C\rangle$, to construct the QCSP $\psi = \langle X \cup \{x\}, Q', D', C \cup \{x = 0\}\rangle$, with $D'_x = \{0, 1\}$. We consider the same reduction and impose that x be placed at the end of the ordered set X. Then x is shallow-fixable to 1 iff it is deep-fixable to 1. We have proved that ϕ is false if variable x is deep-fixable to 1 in ψ, which is true if it is shallow-fixable to 1 in ψ. Similarly in all cases of Prop. 14 the reduction directly applies to shallow property as long as we impose that the new variable x be put at the end of the quantifier prefix. □

PROPOSITION 16. *Given a Σ_k QCSP $\phi = \langle X, Q, D, C\rangle$ encoded using Formalism (I), the problems of deciding whether:*

—*value $a \in D_{x_i}$ is deep-fixable, deep-removable, inconsistent, implied for variable $x_i \in X$,*

—*value $a \in D_{x_i}$ is deep-substitutable to or deep-interchangeable with $b \in D_{x_i}$ for variable $x_i \in X$,*

—*variable $x_i \in X$ is dependent on variables $V \subseteq X$, or is deep-irrelevant,*

are Π_k^p-hard and belong to Π_{k+1}^p. Moreover, for deep inconsistency, implication, determinacy and dependence, the problems are more precisely Π_k^p-complete.

The use of formalism (1) means that ϕ is a Quantified Boolean Formula of the form:

$$\phi : \quad \exists M_1.\ \forall M_2. \cdots Q_k M_k.\ C$$

where the M_is are blocks of variables of alternating quantification, C is a Boolean circuit built on these variables, and the last block M_k is quantified universally ($Q_k = \forall$) if k is even, and existentially ($Q_k = \exists$) if k is odd. Consistently with previous notation, the linearly ordered set $X = \{x_1 \ldots x_n\}$ denotes the union of all variables of the prefix, and the notations E_j, A_j, etc., are defined as in Section 2.1.

For technical reasons it is more convenient to analyze the complexity of the *negations* of these properties, i.e., we focus on the complexity of determining whether the property *does not hold*. So we prove that the negations are Σ_k^p-hard and belong to Σ_{k+1}^p. (The problem of testing whether a Σ_k^pQBF is false is Π_k^p-complete.)

PROOF. (membership results) For **consistency**, membership in Σ_k^p is shown as follows: we are given a formula ϕ of the aforementioned form, as well as a and x_i, and we want to test whether $\exists t \in \mathsf{out}^\phi.\ t_{x_i} = a$. We use a reduction similar to the one used by Prop. 13, and construct a formula which is true iff the property holds. The formula used in Prop. 13 imposes additional constraints whose role is to make sure that the outcome belongs to the set of scenarios of any winning strategy of the produced formula. In our case the outcome in question is quantified existentially and is of the form $\langle v_1, \ldots, v_n\rangle$ with $v_i = a$. We obtain the formula:

$$\psi : \quad \exists v_1, \ldots v_n.\ \exists M_1.\ \forall M_2 \cdots Q_k M_k.\ (B \wedge C \wedge v_i = a) \tag{5}$$

where each variable v_i ranges over D_{x_i} and B is the conjunction:

$$\bigwedge_{x_i \in E} \left(\left(\bigwedge_{y_j \in A_{i-1}} y_j = v_j \right) \rightarrow (x_i = v_i) \right)$$

Note that the existentially quantified variables $\langle v_1, \ldots, v_n \rangle$ are not redundant with the x_js: we want to impose that *at least* one of the outcomes of ψ assign x_i to a, whereas simply adding the constraint $x_i = a$ would enforce it for every scenario of any strategy. Formula ψ is true iff there exists a tuple $t \in \mathsf{out}^\phi$ such that $t_{x_i} = a$ is a direct consequence of Prop. 13. Formula ψ is itself a Σ_k-QBF and we can therefore determine whether it is true in Σ_k^p.

Non-**implication** ($\exists t \in \mathsf{out}.\ t_{x_i} \neq a$), Eq. 5 is simply replaced by:

$$\psi: \quad \exists v_1, \ldots v_n.\ \exists M_1.\ \forall M_2 \cdots Q_k M_k.\ (B \wedge C \wedge \boxed{v_i \neq a})$$

Non-**determinacy** is expressed as $\exists t \in \mathsf{out}.\ \exists b \neq t_{x_i}.\ t[x_i := b] \in \mathsf{out}$ or, equivalently, as $\exists t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ t'|_{X \setminus \{x_i\}} = t|_{X \setminus \{x_i\}} \wedge t'_{x_i} \neq t_{x_i}$. We have to assert the joint existence of the two outcomes t and t', whose values on variables $x_1 \ldots x_n$ are noted $\langle v_1, \ldots, v_n \rangle$ and $\langle v'_1, \ldots, v'_n \rangle$, respectively. We obtain:

$$\psi: \quad \exists v_1, \ldots v_n.\ \exists v'_1, \ldots v'_n.\ \left(\begin{array}{l} \bigwedge_{j \neq i} v'_j = v_j \quad \wedge \quad v'_i \neq v_i \\ \wedge\ \exists M_1.\ \forall M_2 \cdots Q_k M_k.\ (B \wedge C) \\ \wedge\ \exists M'_1.\ \forall M'_2 \cdots Q'_k M'_k.\ (B' \wedge C') \end{array} \right)$$

Now we note that the two matrices $(B \wedge C)$ and $(B' \wedge C')$ are imposed on disjoint sets of variables (the unprimed and the primes variables, respectively), so we can rewrite the previous formula in a Σ_k form, as follows:

$$\psi: \quad \exists v_1, \ldots v_n.\ \exists v'_1, \ldots v'_n.\ \left(\begin{array}{l} \bigwedge_{j \neq i} v'_j = v_j \quad \wedge \quad v'_i \neq v_i \quad \wedge \\ \exists M_1, M'_1.\ \forall M_2, M'_2 \\ \qquad \cdots Q_k M_k, M'_k.\ (B \wedge C \wedge B' \wedge C') \end{array} \right)$$

Non-**dependence** can be stated as $\exists t \in \mathsf{out}.\ \exists t' \in \mathsf{out}.\ (t|_V = t'|_V) \wedge (t_{x_i} \neq t'_{x_i})$, relying on the fact that the domain only has two values; the proof is similar except that ψ has the following form:

$$\psi: \quad \exists v_1, \ldots v_n.\ \exists v'_1, \ldots v'_n.\ \left(\begin{array}{l} \boxed{\bigwedge_{x_j \in V} v'_j = v_j} \quad \wedge \quad v'_i \neq v_i \quad \wedge \\ \exists M_1, M'_1.\ \forall M_2, M'_2 \\ \qquad \cdots Q_k M_k, M'_k.\ (B \wedge C \wedge B' \wedge C') \end{array} \right)$$

For the other properties it is less obvious to see whether the upper bound of Σ_k^p holds, because their negations are defined as follows:

—Non-**fixability** can be expressed as $\exists t \in \mathsf{out}.\ t[x_i := a] \notin \mathsf{out}$ or, equivalently, $\exists t \in \mathsf{out}.\ \exists t' \notin \mathsf{out}.\ t'|_{X \setminus \{x_i\}} = t|_{X \setminus \{x_i\}} \wedge t'_{x_i} = a$;

—Non-**substitutability** as $\exists t \in \mathsf{out}.\ (t_{x_i} = a) \wedge (t[x_i := b] \notin \mathsf{out})$ or, equivalently, $\exists t \in \mathsf{out}.\ \exists t' \notin \mathsf{out}.\ t'|_{X \setminus \{x_i\}} = t|_{X \setminus \{x_i\}} \wedge t_{x_i} = a \wedge t'_{x_i} = b$;

—Non-**removability** as $\exists t \in \mathsf{out}.\ (t_{x_i} = a) \wedge (\forall b \neq a.\ t[x_i := b] \notin \mathsf{out})$ or, equivalently, as $\exists t \in \mathsf{out}.\ \forall t' \notin \mathsf{out}.\ (t'|_{X \setminus \{x_i\}} = t|_{X \setminus \{x_i\}}) \rightarrow t_{x_i} = t'_{x_i}$;

—Non-**irrelevance** is expressed as $\exists t \in \mathsf{out}.\ \exists t' \notin \mathsf{out}.\ t'|_{X \setminus \{x_i\}} = t|_{X \setminus \{x_i\}}$.

The problem is that in each case we need to find both an outcome t and another tuple t' which is not an outcome. The quantifier pattern for asserting that t is not an outcome is now of the form $\forall M_1.\ \exists M_2 \cdots \overline{Q_k} M_k$, where $\overline{Q_k}$ is the dual quantifier to Q_k. For instance for non-irrelevance the obtained formula has the following form:

$$\exists v_1, \dots v_n.\ \exists v'_1, \dots v'_n.\ \begin{pmatrix} \bigwedge_{j\neq i} v'_j = v_j \\ \wedge\ \exists M_1.\ \forall M_2 \cdots Q_k M_k.\ (B \wedge C) \\ \wedge\ \forall M'_1.\ \exists M'_2 \cdots \overline{Q_k}_k M'_k.\ (\neg B' \vee \neg C') \end{pmatrix}$$

Similarly to before, the variables involved in the matrices $(B\wedge C)$ and $(\neg B' \vee \neg C')$ are disjoint and we can merge them into one prefix. We rename the indexing of the primed blocks as follows:

$$\exists v_1, \dots v_n.\ \exists v'_1, \dots v'_n.\ \begin{pmatrix} \bigwedge_{j\neq i} v'_j = v_j \\ \wedge\ \exists M_1.\ \forall M_2 \cdots Q_k M_k.\ (B \wedge C) \\ \wedge\ \forall M'_2.\ \exists M'_3 \cdots Q_{k+1} M'_{k+1}.\ (\neg B' \vee \neg C') \end{pmatrix}$$

and obtain:

$$\exists v_1, \dots v_n.\ \exists v'_1, \dots v'_n.\ \begin{pmatrix} \bigwedge_{j\neq i} v'_j = v_j \ \wedge \\ \exists M_1.\ \forall M_2, M'_2.\ \exists M_3, M'_3. \\ \quad \cdots Q_k M_k.\ Q_{k+1} M_{k+1}\ ((B \wedge C) \wedge (\neg B' \vee \neg C')) \end{pmatrix}$$

which is in Σ^p_{k+1} form. We obtain a similar formula with minor changes for fixability, substitutability and removability. □

Proof. (hardness) The hardness part is easy, because in all the reductions used in Prop. 14 to show the PSPACE-hardness of the properties, we reduced the problem of determining whether a QCSP ϕ is false to the problem of checking the considered property for a new formula ψ. The new formula ψ was constructed by introducing a new existential variable and this variable could be added into any quantifier block. Because of that, we can always make sure that the quantifier prefix of ψ follows the same alternation as the one of ϕ, and we can therefore reduce the problem of determining whether a Σ^p_kQBF is false to the problem of testing the considered property is verified by a Σ^p_kQBF.

For instance the reduction used to prove that inconsistency is PSPACE-complete was as follows: we reduced any QCSP $\phi:\ \exists M_1.\ \forall M_2.\ \cdots Q_k M_k.\ C$ to the QCSP $\psi:\ \exists M'_1, \{x\}.\ \forall M_2.\ \cdots Q_k M_k.\ C$ with $D_x = \{a\}$. We had not specified the precise existential block in which the new variable x was added because the proof was precisely independent of that. We can now impose that it be inserted in the first block M_1. This shows that we can reduce the problem of falsity for Σ^p_kQBFs to the problem of inconsistency for Σ^p_kQBFs. Similarly, all the other proofs can be directly adapted to bounded quantifier alternations. □

Proposition 17. *Let $\phi = \langle X, Q, D, C\rangle$ be a QCSP where $C = \{c_1, \dots, c_m\}$. We denote by ϕ_k the QCSP $\langle X, Q, D, \{c_k\}\rangle$ in which only the k-th constraint is considered. We have, for all $x_i \in X$, $V \subseteq X$, and $a, b \in D_{x_i}$:*

—$\left(\bigvee_{k\in 1..m} \textit{inconsistent}^{\phi_k}(x_i, a) \right) \rightarrow \textit{inconsistent}^{\phi}(x_i, a)$;

—$\left(\bigvee_{k\in 1..m} \textit{implied}^{\phi_k}(x_i, a) \right) \rightarrow \textit{implied}^{\phi}(x_i, a)$;

—$\left(\bigwedge_{k\in 1..m} \textit{d-fixable}^{\phi_k}(x_i, a) \right) \rightarrow \textit{d-fixable}^{\phi}(x_i, a)$;

—$\big(\bigwedge_{k\in 1..m}$ *d-substitutable*$^{\phi_k}(x_i,a,b) \big) \to$ *d-substitutable*$^{\phi}(x_i,a,b)$;

—$\big(\bigwedge_{k\in 1..m}$ *d-interchangeable*$^{\phi_k}(x_i,a,b) \big) \to$ *d-interchangeable*$^{\phi}(x_i,a,b)$;

—$\big(\bigvee_{k\in 1..m}$ *determined*$^{\phi_k}(x_i) \big) \to$ *determined*$^{\phi}(x_i)$;

—$\big(\bigwedge_{k\in 1..m}$ *d-irrelevant*$^{\phi_k}(x_i) \big) \to$ *d-irrelevant*$^{\phi}(x_i)$;

—$\big(\bigvee_{k\in 1..m}$ *dependent*$^{\phi_k}(V,x_i) \big) \to$ *dependent*$^{\phi}(V,x_i)$.

Proof. These propositions rely on the following *monotonicity* property of the set of outcomes: if we have two QCSPs $\phi_1 = \langle X,Q,D,C_1\rangle$ and $\phi_2 = \langle X,Q,D,C_2\rangle$ (with the same quantifier prefix) and if $\mathsf{sol}^{\phi_1} \subseteq \mathsf{sol}^{\phi_2}$ then $\mathsf{out}^{\phi_1} \subseteq \mathsf{out}^{\phi_2}$. This is easy to see: any winning strategy s for ϕ_1 is such that $\mathsf{sce}(s) \subseteq \mathsf{sol}^{\phi_1}$. Then it is also such that $\mathsf{sce}(s) \subseteq \mathsf{sol}^{\phi_2}$ and it is a winning strategy for ϕ_2.

The proofs for inconsistency, implication and determinacy directly follow:

—For inconsistency: if for some k we have $\forall t \in \mathsf{out}^{\phi_k}.\ t_{x_i} \neq a$, then we also have $\forall t \in \mathsf{out}^{\phi}.\ t_{x_i} \neq a$, because $\mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$.

—For implication: if for some k we have $\forall t \in \mathsf{out}^{\phi_k}.\ t_{x_i} = a$, then we also have $\forall t \in \mathsf{out}^{\phi}.\ t_{x_i} = a$, because $\mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$.

—For determinacy: if for some k we have $\forall t \in \mathsf{out}^{\phi_k}.\ \forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{out}^{\phi_k}$, then we also have $\forall t \in \mathsf{out}^{\phi}.\ \forall b \neq t_{x_i}.\ t[x_i := b] \notin \mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$.

—For dependence: if for some k we have $\forall t,t' \in \mathsf{out}^{\phi_k}.\ t|_V = t'|_V \to\ t_{x_i} = t'_{x_i}$, then we also have $\forall t,t' \in \mathsf{out}^{\phi}.\ t|_V = t'|_V \to\ t_{x_i} = t'_{x_i}$ because $\mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$.

Consider now deep fixability. We assume that forall k and forall $t \in \mathsf{out}^{\phi_k}$ we have $t[x_i := a] \in \mathsf{out}^{\phi_k}$. We consider a tuple $t \in \mathsf{out}^{\phi}$; since $\mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$ for all k, t belongs to every out^{ϕ_k}, and therefore $t[x_i := a]$ belongs to every out^{ϕ_k} and therefore to every sol^{ϕ_k}. We conclude that $t[x_i := a] \in \mathsf{sol}^{\phi} = \bigcap_k \mathsf{sol}^{\phi_k}$. We have seen in Prop. 2 that deep fixability can be stated as $\forall t \in \mathsf{out}^{\phi}.\ t[x_i := a] \in \mathsf{sol}^{\phi}$, which completes the proof.

For deep substitutability. We assume that forall k and forall $t \in \mathsf{out}^{\phi_k}$ we have $t_{x_i} = a \to\ t[x_i := b] \in \mathsf{out}^{\phi_k}$. We consider a tuple $t \in \mathsf{out}^{\phi}$ such that $t_{x_i} = a$; since $\mathsf{out}^{\phi} \subseteq \mathsf{out}^{\phi_k}$ for all k, t belongs to every out^{ϕ_k}, and therefore $t[x_i := b]$ belongs to every out^{ϕ_k} and therefore to every sol^{ϕ_k}. We conclude that $t[x_i := b] \in \mathsf{sol}^{\phi} = \bigcap_k \mathsf{sol}^{\phi_k}$. We have seen in Prop. 2 that deep substitutability can be stated as $\forall t \in \mathsf{out}^{\phi}.\ t_{x_i} = a \to t[x_i := b] \in \mathsf{sol}^{\phi}$, which completes the proof.

For deep interchangeability the result follows since two values a and b are interchangeable iff a is substitutable to b and b is substitutable to a.

For deep irrelevance we use a result of Prop. 5: variable x_i is irrelevant iff it is fixable to any value $a \in D_{x_i}$. If forall k we have *d-irrelevant*$^{\phi_k}(x_i)$ then we have, forall k and forall $a \in D_{x_i}$, *d-fixable*$^{\phi_k}(x_i,a)$. It follows that, forall $a \in D_{x_i}$, *d-fixable*$^{\phi}(x_i,a)$. This is equivalent to *d-irrelevant*$^{\phi}(x_i)$. $\Box$