# Belief Merging without Distance Measures

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**Abstract**

When information comes from different sources inconsistent beliefs may appear. To handle inconsistency, several model-based belief merging operators have been proposed. Starting from the beliefs of a group of agents which might conflict, these operators return a unique consistent belief base which represents the beliefs of the group. The operators, parameterized by a distance between interpretations and aggregation function, usually only take into account consistent bases. Consequently some information which is not responsible for conflicts may be ignored. This paper presents *PS-Merge*, an alternative way of merging which is based on the notion of Partial Satisfiability. The proposal uses an alternative way of measuring the satisfaction of a formula since Partial Satisfiability lets us have satisfaction values in the interval [0,1]. *PS-Merge* produces similar results to other merging approaches. Actually, in order to achieve satisfactory results for different scenarios from the literature we require different merging operators while the proposal obtains similar results for all these different scenarios with a unique operator, *PS-Merge*.

## 1 Introduction

Belief merging is concerned with the process of combining the information contained in a set of (possibly inconsistent) belief bases obtained from different sources to produce a single consistent belief base. Belief merging is an important issue in artificial intelligence and databases, and its applications are many and diverse [2]. For example, in multiagent systems a merging operator defines the beliefs of a group of agents according to the beliefs of each member of the group. When agents have conflicting beliefs about the "true" state of the world, belief merging can be used to determine the "true" state of the world for the group. Though we consider only belief bases, merging operators can typically be used for merging either beliefs or goals.

Several merging operators have been defined and characterized in a logical way. Among them, model-based merging operators [10, 7, 15, 11] obtain

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a belief base from a set of interpretations with the help of a distance measure on interpretations and an aggregation function. Usually, model-based merging operators only take into account consistent belief bases and consequently some information which is not responsible for conflicts may be ignored. Other merging operators, syntax-based ones [1], are based on the selection of some consistent subsets of the set-theoretic union of the belief bases. This allows for taking inconsistent belief bases into account, but such operators usually do not take into account the frequency of each explicit item of belief. For example, the fact that a formula $\psi$ is believed in a base or in $n$ bases is not considered relevant, which is counter-intuitive.

An alternative method of merging uses the notion of Partial Satisfiability to define $PS\text{-}Merge$, a model-based merging operator which depends on the syntax of the belief bases [3]. The proposal produces similar results to other merging approaches, but while other approaches require many merging operators in order to achieve satisfactory results for different scenarios the proposal obtains similar results for all these different scenarios with a unique operator. It is worth noticing that $PS\text{-}Merge$ is not based on distance measures on interpretations, and takes into account inconsistent bases and the frequency of each explicit item of belief. We study some logical properties satisfied by $PS\text{-}Merge$ and analyze the rational behavior of the operator.

The rest of the paper is organized as follows. After providing some technical preliminaries, Section 3 describes the notion of Partial Satisfiability and the associated merging operator. Section 4 studies some properties satisfied by $PS\text{-}Merge$ in the context of postulates proposed in [7, 8]. In Section 5 we give a comparison of $PS\text{-}Merge$ with other approaches and Section 6 concludes with a discussion of future work.

## 2 Preliminaries

We consider a language $\mathcal{L}$ of propositional logic formed from a finite ordered set $P := \{p_1, p_2, ..., p_n\}$ of atoms in the usual way. And we use the standard terminology of propositional logic except for the definitions given below. A *belief base* $K$ is a finite set of propositional formulas of $\mathcal{L}$ representing the beliefs of an agent (we identify $K$ with the conjunction of its elements).

A *state* or *interpretation* is a function $w$ from $P$ to $\{1, 0\}$, these values are identified with the classical truth values *true* and $false$ respectively. The set of all possible states will be denoted as $\mathcal{W}$ and its elements will be denoted by vectors of the form $(w(p_1), ..., w(p_n))$. A *model* of a propositional formula $Q$ is a state such that $w(Q) = 1$ once $w$ is extended in the usual way over the connectives. For convenience, if $Q$ is a propositional formula or a set of propositional formulas then $\mathcal{P}(Q)$ denotes the set of atoms appearing in $Q$. $|P|$ denotes the cardinality of set $P$. A *literal* is an atom or its negation.

A belief *profile* $E$ denotes the beliefs of agents $K_1, ..., K_m$ that are involved in the merging process. If $Q_{1_i}, ..., Q_{n_i}$ denotes the beliefs in the base $K_i$, then $E = \{\{Q_{1_1}, ..., Q_{n_1}\}, ..., \{Q_{1_m}, ..., Q_{n_m}\}\}$. $E$ is a multiset (bag) of belief bases and thus two agents are allowed to exhibit identical bases.

Two belief profiles $E_1$ and $E_2$ are said to be equivalent, denoted by $E_1 \equiv E_2$, iff there is a bijection $g$ from $E_1$ to $E_2$ such that $K \equiv g(K)$ for every base $K$ in $E_1$. With $\bigwedge E$ we denote the conjunction of the belief bases $K_i \in E$, while $\sqcup$ denotes the multiset union. For every belief profile $E$ and positive integer $n$, $E^n$ denotes the multiset union of $n$ times $E$.

## 3 Partial Satisfiability

In order to define Partial Satisfiability without loss of generality we consider a normalized language so that each belief base is taken as the disjunctive normal form (DNF) of the conjunction of its elements. Thus if $K = \{Q_1, ..., Q_n\}$ is a belief base we will identify this base with $Q_K = DNF(Q_1 \wedge ... \wedge Q_n)$. The DNF of a formula is obtained by replacing $A \leftrightarrow B$ and $A \rightarrow B$ by $(\neg A \vee B) \wedge (\neg B \vee A)$ and $\neg A \vee B$ respectively, applying De Morgan's laws, using the distributivity law, distributing $\vee$ over $\wedge$ and finally eliminating the literals repeated in each conjunct.

**Example 1.** *Given the belief base $K = \{a \rightarrow b, \neg c\}$ it is identified with $Q_K = (\neg a \wedge \neg c) \vee (b \wedge \neg c)$.*

The last part of the construction of the DNF (the minimization by eliminating literals) is important since the number of literals in each conjunct affects the satisfaction degree of the conjunct. We are not applying other logic minimization methods to reduce the size of the DNF expressions since this may affect the intuitive meaning of the formulas. A further analysis of logic equivalence and the results obtained by the Partial Satisfiability is required.

**Definition 1** (Partial Satisfiability). *Let $K$ be a belief base, $w$ any state of $\mathcal{W}$ and $|P| = n$, we define the Partial Satisfiability of $K$ for $w$, denoted as $w_{ps}(Q_K)$, as follows.*

- *If $Q_K := C_1 \wedge ... \wedge C_s$ where $C_i$ are literals then*

$$w_{ps}(Q_K) = max\left\{\sum_{i=1}^{s} \frac{w(C_i)}{s}, \frac{n - |\mathcal{P}(Q_K)|}{2n}\right\}$$

- *If $Q_K := D_1 \vee ... \vee D_r$ where each $D_i$ is a literal or a conjunction of literals then*

$$w_{ps}(Q_K) = max\,\{w_{ps}(D_1), ..., w_{ps}(D_r)\}$$

The intuitive interpretation of Partial Satisfiability is as follows: it is natural to think that if we have the conjunction of two literals and just one is satisfied then we are satisfying 50% of the conjunction. If we generalize this idea we can measure the satisfaction of a conjunction of one or more literals as the sum of the evaluation of them under the interpretation divided by the number of conjuncts. However, the agent's beliefs may consider only some atoms of the language, in that case the agent is not affected by the decision taken over the atoms not appearing in its beliefs. Hence it is indifferent to the evaluation of these atoms, so we interpret this indifference as a partial satisfaction of 50% for each atom not appearing in its beliefs.

On the other hand the agent is interested in satisfying the literals that appear in its beliefs and we interpret this fact by assigning a satisfaction of 100% to each literal verified by the state and 0% to those that are falsified. As we can see the former intuitive idea is reflected in Definition 1 since the literals that appear in the agent beliefs have their classical value and atoms not appearing have a value of just $\frac{1}{2}$.

Finally, if we have a disjunction of conjunctions the intuitive interpretation of the valuation is to obtain the maximum value of the considered conjunctions.

**Example 2.** *The Partial Satisfiability of the belief base of Example 1 given* $P = \{a, b, c\}$ *and* $w = (1, 1, 1)$ *is*

$$w_{ps}(Q_K) = max\left\{max\{\tfrac{w(\neg a)+w(\neg c)}{2}, \tfrac{1}{6}\}, max\{\tfrac{w(b)+w(\neg c)}{2}, \tfrac{1}{6}\}\right\} = \tfrac{1}{2}.$$

Instead of using distance measures as [7, 11, 8, 12] we have proposed the notion of Partial Satisfiability in order to define a new merging operator. The elected states of the merge are those whose values maximize the sum of the Partial Satisfiability of the bases.

**Definition 2.** *Let $E$ be a belief profile obtained from the belief bases $K_1, ...,$ $K_m$, then the Partial Satisfiability Merge of $E$ denoted by $PS\text{-}Merge(E)$ is a mapping from the belief profiles to belief bases such that the set of models of the resulting base is:*

$$\left\{ w \in \mathcal{W} \middle| \sum_{i=1}^{m} w_{ps}(Q_{K_i}) \geq \sum_{i=1}^{m} w'_{ps}(Q_{K_i}) \ \textit{for all} \ w' \in \mathcal{W} \right\}$$

**Example 3.** *We now give a concrete merging example taken from [14]. The author proposes the following scenario: a teacher asks three students which among three languages, SQL, Datalog and $O_2$, they would like to learn. Let $s$, $d$ and $o$ be the propositional letters used to denote the desire to learn SQL, Datalog and $O_2$, respectively, then $P = \{s, d, o\}$. The first student only wants to learn SQL or $O_2$, the second wants to learn only one of Datalog or $O_2$, and the third wants to learn all three languages. So we have $E = \{K_1, K_2, K_3\}$ with $K_1 = \{(s \vee o) \wedge \neg d\}$, $K_2 = \{(\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)\}$, and $K_3 = \{s \wedge d \wedge o\}$.*

In [12] using the Hamming distance applied to the anonymous aggregation function $\Sigma$ and in [7] using the operator $\Delta_{\Sigma}$, both approaches obtain the states $(0,0,1)$ and $(1,0,1)$ as models of the merging.

We have $Q_{K_1} = (s \wedge \neg d) \vee (o \wedge \neg d)$, $Q_{K_2} = (\neg s \wedge d \wedge \neg o) \vee (\neg s \wedge \neg d \wedge o)$, and $Q_{K_3} = s \wedge d \wedge o$. As we can see in the fifth column of Table 1 the models of $PS\text{-}Merge(E)$[1] are the states $(0,0,1)$ and $(1,0,1)$.

| $w$ | $Q_{K_1}$ | $Q_{K_2}$ | $Q_{K_3}$ | $Sum$ | | $min$ |
|---|---|---|---|---|---|---|
| $(1,1,1)$ | $\frac{1}{2}$ | $\frac{1}{3}$ | 1 | $\frac{11}{6} \simeq$ | 1.83 | $\frac{1}{3}$ |
| $(1,1,0)$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{11}{6} \simeq$ | 1.83 | $\frac{1}{2}$ |
| $(\mathbf{1},\mathbf{0},\mathbf{1})$ | 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\mathbf{\frac{14}{6}} \simeq$ | **2.33** | $\mathbf{\frac{2}{3}}$ |
| $(1,0,0)$ | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{10}{6} \simeq$ | 1.67 | $\frac{1}{3}$ |
| $(0,1,1)$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{11}{6} \simeq$ | 1.83 | $\frac{1}{2}$ |
| $(0,1,0)$ | $\frac{1}{6}$ | 1 | $\frac{1}{3}$ | $\frac{9}{6} =$ | 1.5 | $\frac{1}{6}$ |
| $(\mathbf{0},\mathbf{0},\mathbf{1})$ | 1 | 1 | $\frac{1}{3}$ | $\mathbf{\frac{14}{6}} \simeq$ | **2.33** | $\frac{1}{3}$ |
| $(0,0,0)$ | $\frac{1}{2}$ | $\frac{2}{3}$ | 0 | $\frac{7}{6} \simeq$ | 1.16 | 0 |

Table 1: $PS\text{-}Merge$ of Example 3 and min function.

In [8] two classes of merging operators are defined: majority and arbitration merging. The former strives to satisfy a maximum of agents' beliefs and the latter tries to satisfy each agent beliefs to the best possible degree. The former notion is treated in the context of $PS\text{-}Merge$, and it can be refined tending to arbitration if we calculate the minimum value among the Partial Satisfiability of the bases. Then with this indicator, we have a form to choose the state that is impartial and tries to satisfy all agents as far as possible. If we again consider Example 3 in Table 1 there are two different states that maximize the sum of the Partial Satisfaction of the profile, $(1,0,1)$ and $(0,0,1)$. If we try to minimize the individual dissatisfaction these two states do not provide the same results. Using the $min$ function (see $6^{th}$ column of Table 1) over the partial satisfaction of the bases we get the states that minimize the individual dissatisfaction and between the states $(1,0,1)$ and $(0,0,1)$ obtained by the proposal we might prefer the state $(1,0,1)$ over $(0,0,1)$ as the $\Delta_{GMax}$ operator (an arbitration operator) does in [7].

It is possible to extend this notion of $PS\text{-}Merge$ in the case where a set of integrity constraints must be obeyed. If $\mu$ is a formula representing the set of integrity constraints, then the states that falsify the integrity constraint cannot be considered in the $PS\text{-}Merge$. If $\mathcal{W}(\mu)$ denotes the set of states that validate the integrity constraints, it is enough to restrict the definition of the Partial Satisfiability Merge to $\mathcal{W}(\mu)$.

[1] If $\Delta$ is a merging operator, we are going to abuse the notation by referring to the models of the merging operator $mod(\Delta(E))$ and their respective belief base $\Delta(E)$ simply as $\Delta(E)$.

**Definition 3.** *Let $E$ be a belief profile obtained from the belief bases $K_1, ...,$ $K_m$, then PS-Merge$_\mu(E)$, the Partial Satisfiability Merge of $E$ given the set of integrity constraints $\mu$, is a mapping from the belief profiles to belief bases such that the set of models of the resulting base is:*

$$\left\{ w \in \mathcal{W}(\mu) \middle| \sum_{i=1}^{m} w_{ps}(Q_{K_i}) \geq \sum_{i=1}^{m} w'_{ps}(Q_{K_i}) \text{ for all } w' \in \mathcal{W}(\mu) \right\}$$

**Example 4.** *The following example of information merging under constraints is given in [8]. At a meeting of four co-owners of a block of flats, the chairman proposes the construction of a swimming-pool, a tennis-court and a private-car-park in the coming year. But if two of these three items are built, the rent will increase significantly. We will denote by $s$, $t$ and $p$ the construction of the swimming-pool, the tennis-court and the private car-park respectively and $i$ will denote the increase of the rent. Two co-owners want to build the three items, and do not care about the rent increase ($K_1 = K_2 = s \wedge t \wedge p$), the third thinks that building any item will cause at some time an increase of the rent and wants to pay the lowest rent so he is opposed to any construction (so $K_3 = \neg s \wedge \neg t \wedge \neg p \wedge \neg i$) and finally the last one thinks that the flat really needs a tennis-court and a private car-park but does not want a rent increase (i.e. $K_4 = t \wedge p \wedge \neg i$).*

The chairman outlines that building two or more items will increase the rent significantly. This fact cannot be ignored and the states in which this fact is falsified must be ignored. These kinds of facts are known as integrity constraints. In the example the integrity constraints $\mu$ are represented by the single formula $((s \wedge t) \vee (s \wedge p) \vee (t \wedge p)) \rightarrow i$. If we consider $P$ the ordered set $\{s, t, p, i\}$ then the states $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ and $(0, 1, 1, 0)$ cannot be considered as a possible Partial Satisfiability Merge since these states falsify the integrity constraint. It is enough to calculate the Partial-Satisfiability to states in $\mathcal{W}(\mu)$.

The answer to Example 4 obtained by applying *PS-Merge* (see Table 2) is the state $(1, 1, 1, 1)$, i.e. the decision that satisfies the majority of the group is to build the three items no matter if the rent increases. This decision is also the one obtained using the integrity constraint majority merging operator based on the $\Sigma$ function in [8, 9].

# 4 Properties

Finding a set of axiomatic properties that an operator may satisfy in order to exhibit a rational behavior is a concern greatly studied. In [7, 15, 10, 11] sets of postulates have been proposed concerning belief merging operators.

In [7, 9] Konieczny and Pino-Pérez proposed the basic properties (A1)-(A6) for merging operators, rephrased without reference to integrity constraints.

| $w$ | $Q_{K_1}$ | $Q_{K_2}$ | $Q_{K_3}$ | $Q_{K_4}$ | $Sum$ |
|---|---|---|---|---|---|
| $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$ | $1$ | $1$ | $0$ | $\frac{2}{3}$ | $\mathbf{\frac{8}{3}}$ |
| $(1,1,1,0)^*$ | $1$ | $1$ | $\frac{1}{4}$ | $1$ | $\frac{13}{4}$ |
| $(1,1,0,1)$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{23}{12}$ |
| $(1,1,0,0)^*$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{15}{6}$ |
| $(1,0,1,1)$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{23}{12}$ |
| $(1,0,1,0)^*$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{15}{6}$ |
| $(1,0,0,1)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{31}{24}$ |
| $(1,0,0,0)$ | $\frac{1}{3}$ | $\frac{1}{8}$ | $\frac{3}{4}$ | $\frac{1}{3}$ | $\frac{37}{24}$ |
| $(0,1,1,1)$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{4}$ | $\frac{2}{3}$ | $\frac{27}{12}$ |
| $(0,1,1,0)^*$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $1$ | $\frac{17}{6}$ |
| $(0,1,0,1)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{3}{2}$ |
| $(0,1,0,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{25}{12}$ |
| $(0,0,1,1)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{3}{2}$ |
| $(0,0,1,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{3}{4}$ | $\frac{2}{3}$ | $\frac{25}{12}$ |
| $(0,0,0,1)$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{3}{4}$ | $\frac{1}{8}$ | $\frac{9}{8}$ |
| $(0,0,0,0)$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $1$ | $\frac{1}{3}$ | $\frac{19}{12}$ |

Table 2: PS-Merge table of Example 4.

**Definition 4.** *Let $E$, $E_1$, $E_2$ be belief profiles, and $K_1$ and $K_2$ be consistent belief bases. Let $\Delta$ be an operator which assigns to each belief profile $E$ a belief base $\Delta(E)$. $\Delta$ is a merging operator if and only if it satisfies the following postulates:*

*(A1)* $\Delta(E)$ *is consistent*
*(A2) if* $\bigwedge E$ *is consistent then* $\Delta(E) \equiv \bigwedge E$
*(A3) if* $E_1 \equiv E_2$*, then* $\Delta(E_1) \equiv \Delta(E_2)$
*(A4)* $\Delta(\{K_1, K_2\}) \wedge K_1$ *is consistent if and only if* $\Delta(\{K_1, K_2\}) \wedge K_2$ *is consistent*
*(A5)* $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$
*(A6) if* $\Delta(E_1) \wedge \Delta(E_2)$ *is consistent, then* $\Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2)$

The intuitive meaning of the postulates is as follows: (A1) ensures the extraction of a piece of information from the profile. (A2) states that if the belief bases agree on some alternatives, then the result of the merging will be these alternatives. (A3) ensures that the operator obeys a principle of irrelevance of syntax. (A4) is the fairness postulate, such that when we merge two bases the operator should not give preference to one of them. (A5) expresses the following: if we have two groups viewed as profiles $E_1$ and $E_2$, and $E_1$ compromises a set of alternatives to which $A$ belongs, and $E_2$ compromises another set which also contains $A$, then if we join the two groups $A$ must be in the chosen alternatives. (A5) and (A6) together state that if one could find two groups which agree on at least one alternative, then the result of the global merging will be exactly these alternatives.

We analyze the minimal set of properties $PS\text{-}Merge$ satisfies and its rational behavior concerning merging. Clearly $PS\text{-}Merge$ satisfies (A1), which simply requires of the result of merging to be consistent. $PS\text{-}Merge$ also satisfies (A2).

**Proposition 1.** *$\bigwedge E \nvDash \perp$ implies $PS\text{-}Merge(E) \equiv \bigwedge E$.*

*Proof.* Let $E = \{Q_{K_1}, ..., Q_{K_m}\}$ be a profile with its belief bases expressed in DNF such that $\bigwedge E \nvDash \perp$. There are $l > 0$ states $w_1, ..., w_l$ that satisfy each base thus for every state $w_r$ we can find $m$ disjoints $d_1, ..., d_m$ belonging to each base $Q_{K_1}, ..., Q_{K_m}$ respectively, that are satisfied by $w_r$. Consequently the Partial Satisfiability of the bases for every $w_r$ is evaluated in 1, i.e. $w_{rps}(Q_{K_j}) = 1$ for $1 \leq r \leq l$ and $1 \leq j \leq m$. So we can affirm that $\sum_{i=1}^{m} w_{rps}(Q_{K_i}) = m$ for each model of the profile. Notice that every disjoint can have either of two values $\sum_{i=1}^{s} \frac{w(C_i)}{s}$ or $\frac{n-|P(d_j)|}{2n}$ (see Definition 1). Moreover the first value is less or equal to 1 and the second one is less or equal to $\frac{n}{2n} = \frac{1}{2}$. From this fact we can affirm that if a state $w$ does not satisfy a base $Q_K$ then $w_{ps}(Q_K) < 1$ and we can conclude that $\sum_{i=1}^{m} w_{ps}(Q_{K_i}) < m$ for the states that do not satisfy the profile. Hence a state $w$ is included in the merge iff $w$ is a model of $\bigwedge E$, i.e. we obtain only models of the conjunction of the bases as a result of $PS\text{-}Merge$ when the profile is consistent. $\square$

The next property (A3) is a version of Dalal's principle of the Irrelevance of Syntax [4]. In general, $PS\text{-}Merge$ does not satisfy (A3). Consider the situation, called implicit knowledge in [6], where systems want to extract additional knowledge that is not locally held by any agent. For example, if an agent knows $a$ and another agent knows $a \rightarrow b$, then combining their knowledge yields $b$, whereas neither one of them individually knows it. Using most of the merging operators we can find the expected result. Now suppose that this situation is presented in the mind of an agent, i.e. both facts $a$ and $a \rightarrow b$ are known by an agent who does not know how to combine the facts in order to produce $b$ and hence its beliefs in DNF are $K_1 = (a \wedge \neg a) \vee (a \wedge b)$. On the other hand suppose another agent who knows explicitly that $a$ and $b$ hold, i.e. its beliefs in DNF are $K_2 = a \wedge b$. We can see that both agents' bases are equivalent. Now using $PS\text{-}Merge$ to combine the bases with another agent's base $K_3 = \neg b$, we obtain the states $(1, 0)$ and $(0, 0)$ from merging $K_1$ and $K_3$ and only the state $(1, 0)$ from merging $K_2$ and $K_3$. $PS\text{-}Merge$ is a majority operator which tries to satisfy each base as much as possible. Hence in the first case the maximum percentage of satisfaction for $K_1$ is 50% if it wants to leave a percentage of satisfaction for $K_3$ different from 0%, noticing that state $(1, 0)$ satisfies $a$ and $(0, 0)$ satisfies $a \rightarrow b$. In the second case where $K_2$ is satisfied 50% by $(1, 0)$, we can see that if the agent knows explicitly the facts then $PS\text{-}Merge$ refines the answer. We can also see that even though (A3) is not satisfied by $PS\text{-}Merge$, the results show a realistic behavior. The result of combining information without making inferences beforehand might not be as detailed as when agents find some consequences of their knowledge before the merging.

In general, $PS\text{-}Merge$ does not satisfy (A4). Consider again $K_1 = (a \wedge \neg a) \vee (a \wedge b)$ and $K_3 = \neg b$. We can see that both bases are consistent by themselves, however, their conjunction is not. As we know from the example above, using $PS\text{-}Merge$ to combine them we obtain the states $(1, 0)$ and $(0, 0)$ which clearly favor $K_3$. Here it is important to notice that $K_1$ shows an indecision $\neg a \vee b$ of the agent that is why the merging process prefers the satisfaction of the "confident" source $K_3$. However, if $PS\text{-}Merge$ takes as parameters bases showing only explicit information, for example $K_2 = a \wedge b$ and $K_3 = \neg b$, the merging process does not lead to a preference for any of them. The result of the example is state $(1, 0)$ which is not the models of either base.

If there is no "redundant" information, i.e. formulas including disjoints of the style $a \wedge \neg a$, then (A3) and (A4) are satisfied. $PS\text{-}Merge$ satisfies the property (A4) under certain restrictions.

**Proposition 2.** $\Delta(\{K_1, K_2\}) \wedge K_1 \nvDash \perp$ *iff* $\Delta(\{K_1, K_2\}) \wedge K_2 \nvDash \perp$.

(A5) and (A6) establish connections between two results; the result obtained when merging each of two belief profiles and then taking their conjunction and the result obtained when first combining the two belief profiles and then performing a single merge. Together the two properties require that these two results be equivalent, provided that the conjunction referenced is not inconsistent. $PS\text{-}Merge$ satisfies (A5) but it is necessary to consider that profiles come from different contexts and they can have different languages. In this case it will be necessary to extend the language of $E_1$ to include the atoms appearing in $E_2$ and vice versa.

**Proposition 3.** *If* $\mathcal{P}(E_1) = \mathcal{P}(E_2)$ *then* $PS\text{-}Merge(E_1) \wedge PS\text{-}Merge(E_2) \models PS\text{-}Merge(E_1 \sqcup E_2)$

*Proof.* If $PS\text{-}Merge(E_1) \wedge PS\text{-}Merge(E_2)$ is consistent then each model $w$ of the conjunction maximizes the Partial Satisfaction of $E_1$ and $E_2$ at the same time because $w$ is model of each merging. I.e. $\sum_{k_i \in E_1} w_{ps}(Q_{K_i}) \geq \sum_{k_i \in E_1} w'_{ps}(Q_{K_i})$ and $\sum_{k_i \in E_2} w_{ps}(Q_{K_i}) \geq \sum_{k_i \in E_2} w'_{ps}(Q_{K_i})$ for all $w' \in \mathcal{W}$. The merging of the union of the profiles is simply the sum of the Partial Satisfaction of the profiles $E_1$ and $E_2$. Then for all $w' \in \mathcal{W}$:

$$\sum_{k_i \in E_1 \sqcup E_2} w_{ps}(Q_{K_i}) = \sum_{k_i \in E_1} w_{ps}(Q_{K_i}) + \sum_{k_i \in E_2} w_{ps}(Q_{K_i}) \geq \sum_{k_i \in E_1} w'_{ps}(Q_{K_i}) + \sum_{k_i \in E_2} w'_{ps}(Q_{K_i}) = \sum_{k_i \in E_1 \sqcup E_2} w'_{ps}(Q_{K_i})$$

□

**Remark 1.** *By definition the PS-Merge is commutative. I.e. the result of the merging does not depend on any order of the bases of the profile.*

As stated before there are two important classes of merging operators, majority and arbitration operators. The behavior of majority operators is to say that if an opinion is the most popular, then it will be the opinion of the group. A postulate that captures this idea is the postulate (M7) of [7].

*(M7)* $\forall K \exists n \in \mathbb{N} \quad \Delta(E \sqcup \{K\}^n) \models K$

$PS\text{-}Merge$ satisfies the postulate (M7) as a direct consequence of the definition of $PS\text{-}Merge$. Even more, the definition of $PS\text{-}Merge$ not only tries to satisfy the majority of the group, it also tries to satisfy to the maximum degree (see for example 10 in the following section). $PS\text{-}Merge$ does not satisfy all the postulates (A1)-(A6), however, it behaves as a majority merging operator. As the reader can see in the next section the behavior of $PS\text{-}Merge$ is close to $\Delta_\Sigma$ and $CMerge$ which are majority operators.

# 5 Comparing results

$PS\text{-}Merge$ yields similar results compared with existing techniques such as $CMerge$, the $\Delta_\Sigma$[2] operator and $MCS$ (Maximal Consistent Subsets) considered in [11, 7, 8]. Let $E$ be in each case the belief profile consisting of the belief bases enlisted below and let $P$ be corresponding set of atoms ordered alphabetically.

1. $K_1 = K_2 = \{a\}$ and $K_3 = \{\neg a\}$. $CMerge(E) = \{a\}$ which is equivalent to $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1)\}$.

2. $K_1 = \{b\}$, $K_2 = \{a, a \to b\}$ and $K_3 = \{\neg b\}$. Here $CMerge(E) = \{a, a \to b\}$ which is equivalent to $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,1)\}$.

3. $K_1 = \{b\}$, $K_2 = \{a, b\}$ and $K_3 = \{\neg b\}$. In this case $CMerge(E)$ and the model obtained from $\Delta_\Sigma(E)$ and $PS\text{-}Merge(E)$ are as in the previous case.

4. $K_1 = \{b\}$, $K_2 = K_3 = \{a \to b\}$ and $K_4 = \{a, \neg b\}$. $CMerge(E) = \{a, a \to b\}$ and $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,1)\}$ which are all equivalent.

5. $K_1 = \{a, c\}$, $K_2 = \{a \to b, \neg c\}$ and $K_3 = \{b \to d, c\}$. In this case $CMerge(E) = \{a, a \to b, b \to d, c\}$ which is equivalent to $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,1,1,1)\}$.

[2]As stated in [7], merging operator $\Delta_\Sigma$ is equivalent to the merging operator proposed by Lin and Mendelzon in [11] called $CMerge$.

6. $K_1 = \{a, c\}$, $K_2 = \{a \rightarrow b, \neg c\}$, $K_3 = \{b \rightarrow d, c\}$ and $K_4 = \{\neg c\}$. While $CMerge(E)$ $=MCS(E)$ $=\{a, a \rightarrow b, b \rightarrow d\}$ which is equivalent to $\Delta_\Sigma(E) = \{(1,1,0,1),(1,1,1,1)\}$, $PS\text{-}Merge(E) = \{(1,1,0,1)\}$. $CMerge$, $MCS$ and the $\Delta_\Sigma$ operator give no information about $c$. Using $PS\text{-}Merge$, $c$ is falsified and this leads us to have total satisfaction of the second and fourth bases and partial satisfaction of the first and third bases.

7. $K_1 = \{a\}$, $K_2 = \{a \rightarrow b\}$ and $K_3 = \{a, \neg b\}$. Now $CMerge(E) = \{a\}$, $\Delta_\Sigma(E) = \{(1,1),(1,0)\}$ and $PS\text{-}Merge(E) = \{(1,1)\}$. The model $(1,0)$ satisfies only two bases while the model $(1,1)$ satisfy two bases and a “half” of the third base.

8. $K_1 = \{a\}$, $K_2 = \{a \rightarrow b\}$, $K_3 = \{a, \neg b\}$ and $K_4 = \{\neg b\}$. In this case $CMerge(E) = \{a \wedge \neg b\}$, which is equivalent to $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,0)\}$.

9. $K_1 = \{b\}$, $K_2 = \{a \rightarrow b\}$ and $K_3 = \{a, \neg b\}$. Now $CMerge(E) = \{a \wedge b\}$ and $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,1)\}$.

10. $K_1 = \{b\}$, $K_2 = \{a \rightarrow b\}$, $K_3 = \{a, \neg b\}$ and $K_4 = \{\neg b\}$. In this case $CMerge(E) = \{a \vee \neg b\}$, $\Delta_\Sigma(E) = \{(0,0),(1,0),(1,1)\}$ and $PS\text{-}Merge(E) = \{(1,1),(0,0)\}$. The model $(1,0)$ obtained using $\Delta_\Sigma$ operator satisfies only two bases, while the two options of $PS\text{-}Merge(E)$ satisfy two bases and a “half” of the third base. Then $PS\text{-}Merge$ is a refinement of the answer given by $CMerge$ and $\Delta_\Sigma$.

11. $K_1 = K_2 = \{a \wedge b \wedge c\}$, $K_3 = \{\neg a \wedge \neg b \wedge \neg c \wedge \neg d\}$ and $K_4 = \{b \wedge c \wedge \neg d\}$ with the restriction that if two of $a$, $b$ or $c$ are validated it forces $d$ to be validated as well. $CMerge(E) = \{a \wedge b \wedge c \wedge d\}$, $PS\text{-}Merge(E) = \Delta_\Sigma(E) = \{(1,1,1,1)\}$.

# 6 Conclusion

A merging operator has been proposed in [3] that is not defined in terms of a distance measure on interpretations, but is Partial Satisfiability-based. It appears to resolve conflicts among the belief bases in a natural way. The idea is intended to extend the notion of satisfiability to one that includes a “measure” of satisfaction. This notion of satisfaction considers that whenever an atom does not appear in a formula then it is considered that the agent has no preferences on this atom so a partial satisfaction different from 0 is assigned. In Definition 1 we chose $\frac{1}{2}$. This measure considers the intuitive idea that an “or” is satisfied if any of its disjoints is satisfied and in the case of an “and” we count the number of conjuncts satisfied; but if none then we count the partial satisfaction of the atoms not appearing in the conjunction. We can think that a state always satisfies a formula by a percentage, which is given by the Partial Satisfiability. Once a satisfaction measure of belief bases is given, it is used to define $PS\text{-}Merge$. Unlike the operators proposed in the literature, in order to know the "degree" of satisfaction by a given state, $PS\text{-}Merge$ does not need to calculate a partial pre-order over the set of states since Partial Satisfiability can be calculated for a single state. In this way the comparison between states becomes easier. This property can be used in many real-world collective decision problems, as a set of alternatives is given and the method selects a collectively preferred belief base from the set of candidates. However, it is necessary to take into account that before calculating the Partial Satisfiability of a formula it is necessary to transform it into DNF.

Unlike other approaches $PS\text{-}Merge$ can consider belief bases which are inconsistent, since the source of inconsistency can refer to specific atoms and the operator takes into account the rest of the information.

The approach bears some resemblance to the belief merging framework proposed in [7, 8, 11, 12], particularly with the $\Delta_{\Sigma}$ operator. As with those approaches the $Sum$ function is used, but instead of using it to measure the distance between the states and the profile $PS\text{-}Merge$ uses $Sum$ to calculate the general degree of satisfiability. The result of $PS\text{-}Merge$ are simply the states which maximize the $Sum$ of the Partial Satisfiability of the profile and it is not necessary to define a partial pre-order. Because of this similarity between $PS\text{-}Merge$ and $\Delta_{\Sigma}$ we propose to analyze this similarity in term of the postulates satisfied by $\Delta_{\Sigma}$ outlined in [7, 8]. In this paper we analyzed some of these postulates, and even though the $PS\text{-}Merge$ does not satisfy all the properties cited in [7, 11] it has a rational behavior.

As in [8] in order to consider integrity constraints $PS\text{-}Merge$ selects the states among the states which validate the integrity constraints rather than those in $\mathcal{W}$. The approach behaves as a majority operator but an arbitration operator can also be defined in terms of Partial Satisfiability in a similar way.

As future work a further analysis of the $PS\text{-}Merge$ is necessary to characterize its behaviour in terms of postulates. As well, study of the properties of the approach including integrity constraints is required. It remains for the definition of an arbitration operator in terms of Partial Satisfiability and the corresponding characterization to be considered. It is necessary to study the complexity of the whole process of the $PS\text{-}Merge$ in order to compare it with the existing techniques. Finally, we intend to combine the proposal with a heuristic for solving problems with combinatorial explosion.

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